

# DISTORTION AND COVERING THEOREMS OF PLURIHARMONIC MAPPINGS

SH. CHEN AND S. PONNUSAMY <sup>†</sup>

ABSTRACT. In this paper, we mainly investigate distortion and covering theorems on some classes of pluriharmonic mappings.

## 1. INTRODUCTION AND PRELIMINARIES

The notion of linear-invariant family (hereafter  $\mathcal{LIF}$ ) of holomorphic functions defined on the unit disk  $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$  was first introduced by Pommerenke in [19] and showed a number of important properties of such families. Recall that if  $\mathcal{A}$  denotes the family of all holomorphic functions  $f$  on  $\mathbb{D}$  with the topology of uniform convergence of compact subsets of  $\mathbb{D}$ , then a subfamily  $\mathcal{F}$  of  $\mathcal{A}$  is called linear-invariant if it is closed under the re-normalized composition with a conformal automorphism of  $\mathbb{D}$ . If the modulus of the second Taylor coefficient is bounded in  $\mathcal{F}$ , then the order  $\alpha$  of the  $\mathcal{LIF}$  is defined to be

$$\alpha := \sup\{|f''(0)|/2 : f \in \mathcal{F}\}.$$

Many properties of a  $\mathcal{LIF}$  depends on the order of the family. A universal  $\mathcal{LIF}$  of order  $\alpha$ , denoted by  $\mathcal{U}_\alpha$ , is the union of all  $\mathcal{LIF}$ 's  $\mathcal{F}$  such that order of  $\mathcal{F}$  less than or equal to  $\alpha$ . The fact is that  $\mathcal{U}_\alpha$  is empty if  $\alpha < 1$  and  $\mathcal{U}_1$  coincides with the family of all normalized holomorphic functions  $f$  which univalently map  $\mathbb{D}$  onto convex domains, see [19]. Also, a  $\mathcal{LIF}$  of order 2 is the family  $\mathcal{S}$  of normalized univalent functions from  $\mathcal{A}$ . Moreover, it has been proved that many subfamilies of univalent mappings on  $\mathbb{D}$  are linearly invariant, see for example [13] and the references therein. For the regularity growth of functions on  $\mathcal{U}_\alpha$ , we refer to [2, 21, 22]. The concept of linear invariance was generalized by many authors in many different contexts and in 1997, Pfaltzgraff [16] extended this concept for locally holomorphic functions defined on the unit ball of the complex Euclidean  $n$ -space  $\mathbb{C}^n$  and many properties were further discussed in [17, 18]. For our discussion, we need to deal with such problems in the higher dimensional case.

As with the standard practice, for  $z = (z_1 \cdots z_n)$  and  $w = (w_1 \cdots w_n)$  in  $\mathbb{C}^n$ , we let  $\bar{z} = (\bar{z}_1 \cdots \bar{z}_n)$ , and  $\langle z, w \rangle := \sum_{k=1}^n z_k \bar{w}_k$  with the associated Euclidean norm  $\|z\| := \langle z, z \rangle^{1/2}$  which makes  $\mathbb{C}^n$  into an  $n$ -dimensional complex Hilbert space.

---

File: Plur-map\_final.tex, printed: 7-10-2014, 1.40

2000 *Mathematics Subject Classification*. Primary: 30C99, 31C10; Secondary: 15A04, 30C62, 32A10.

*Key words and phrases*. Pluriharmonic mapping, distortion theorem, linear invariant family, quasiregular mapping.

<sup>†</sup> Corresponding author.

Throughout the discussion an element  $z \in \mathbb{C}^n$  is identified as an  $n \times 1$  column vector. For  $a \in \mathbb{C}^n$  and  $r > 0$ ,

$$\mathbb{B}^n(a, r) = \{z \in \mathbb{C}^n : \|z - a\| < r\}$$

denotes the (open) ball of radius  $r$  with center  $a$ . Also, we let  $\mathbb{B}^n(r) := \mathbb{B}^n(0, r)$  and use  $\mathbb{B}^n$  to denote the unit ball  $\mathbb{B}^n(1)$ , and  $\mathbb{D} = \mathbb{B}^1$ .

A continuous complex-valued function  $f$  defined on a domain  $G \subset \mathbb{C}^n$  is said to be *pluriharmonic* if for each fixed  $z \in G$  and  $\theta \in \partial\mathbb{B}^n$ , the function  $f(z + \theta\zeta)$  is harmonic in  $\{\zeta \in \mathbb{C} : \|\theta\zeta - z\| < d_G(z)\}$ , where  $d_G(z)$  denotes the distance from  $z$  to the boundary  $\partial G$  of  $G$ . It follows from [20, Theorem 4.4.9] that a real-valued function  $u$  defined on  $G$  is pluriharmonic if and only if it is locally the real part of a holomorphic function. If  $\Omega$  is a simply connected domain in  $\mathbb{C}^n$ , then it is clear that a mapping  $f : \Omega \rightarrow \mathbb{C}$  is pluriharmonic if and only if  $f$  has a representation  $f = h + \bar{g}$ , where  $h, g$  are holomorphic in  $\Omega$  (cf. [25]). A *vector-valued mapping*  $f = (f_1 \cdots f_N)^T$  defined in  $\mathbb{B}^n$  is said to be pluriharmonic, if each component  $f_j$  ( $1 \leq j \leq N$ ) is a pluriharmonic mapping from  $\mathbb{B}^n$  into  $\mathbb{C}$ , where  $N$  is a positive integer and  $T$  is the transpose of a matrix. We refer to [4, 6, 7, 11, 12, 20] for further details and recent investigations on pluriharmonic mappings.

For an  $n \times n$  complex matrix  $A$ , we introduce the *operator norm*

$$\|A\| = \sup_{z \neq 0} \frac{\|Az\|}{\|z\|} = \max \{\|A\theta\| : \theta \in \partial\mathbb{B}^n\}.$$

We use  $L(\mathbb{C}^n, \mathbb{C}^m)$  to denote the space of continuous *linear operators* from  $\mathbb{C}^n$  into  $\mathbb{C}^m$  with the operator norm, and let  $I_n$  be the *identity operator* in  $L(\mathbb{C}^n, \mathbb{C}^n)$ .

We denote by  $\mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$  the set of all *vector-valued pluriharmonic mappings* from  $\mathbb{B}^n$  into  $\mathbb{C}^n$ . Then every  $f \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$  can be written as  $f = h + \bar{g}$ , where  $h$  and  $g$  are holomorphic in  $\mathbb{B}^n$ , and this representation is unique when  $g(0) = 0$ . It is a simple exercise to see that the real Jacobian determinant of  $f$  can be written as

$$\det J_f = \det \begin{pmatrix} Dh & \overline{Dg} \\ Dg & \overline{Dh} \end{pmatrix}$$

and if  $h$  is locally biholomorphic (i.e. the complex Jacobian matrix  $J_f(z)$  of  $f$  at each  $z$  is invertible), then the determinant of  $J_f$  has the form

$$(1.1) \quad \det J_f = |\det Dh|^2 \det \left( I_n - Dg[Dh]^{-1} \overline{Dg[Dh]^{-1}} \right).$$

In the case of a *planar harmonic mapping*  $f = h + \bar{g}$ , we find that

$$\det J_f = |h'|^2 - |g'|^2,$$

and so,  $f$  is locally univalent and sense-preserving in  $\mathbb{D}$  if and only if  $|g'(z)| < |h'(z)|$  in  $\mathbb{D}$ ; or equivalently if  $h'(z) \neq 0$  and the dilatation  $\omega(z) = g'(z)/h'(z)$  is analytic in  $\mathbb{D}$  and has the property that  $|\omega(z)| < 1$  in  $\mathbb{D}$  (see [10, 14]). For  $f = h + \bar{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$ , the condition  $\|Dg[Dh]^{-1}\| < 1$  is sufficient for  $\det J_f$  to be positive and hence for  $f$  to be sense-preserving. This is indeed a natural generalization of one-variable condition (cf. [11]).

For motivation, consider the Taylor expansion of a function  $f = h + \bar{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$  with  $h(0) = g(0) = 0$ , where

$$(1.2) \quad \begin{aligned} h(z) &= [Dh(0)]z + \frac{1}{2}[D^2h(0)](z, z) + \cdots + \frac{1}{m}[D^mh(0)](z, \dots, z) + \cdots \\ &= A_1z + A_2(z, z) + A_m(z, \dots, z) + \cdots \end{aligned}$$

and

$$(1.3) \quad \begin{aligned} g(z) &= [Dg(0)]z + \frac{1}{2}[D^2g(0)](z, z) + \cdots + \frac{1}{m}[D^mg(0)](z, \dots, z) + \cdots \\ &= B_1z + B_2(z, z) + B_m(z, \dots, z) + \cdots \end{aligned}$$

As with one variable case, a  $\mathcal{LIF}$  in  $\mathbb{B}^n$  is a family  $\mathcal{M}$  of locally biholomorphic mappings  $f : \mathbb{B}^n \rightarrow \mathbb{C}^n$  such that if  $f \in \mathcal{M}$  then

- (i)  $f(0) = 0$ ,  $Df(0) = I_n$  and
- (ii)  $\Lambda_\phi(f) \in \mathcal{M}$  for all  $\phi \in \text{Aut}(\mathbb{B}^n)$ , the holomorphic automorphism of  $\mathbb{B}^n$ .

Here  $\Lambda_\phi(f) = [D\phi(0)]^{-1}[Df(\phi(0))]^{-1}[f(\phi(z)) - f(\phi(0))]$  denotes the *Koebe transform* of  $f$  (cf. [17, 18]) and thus, the classical definition of the order  $\alpha$  of  $\mathcal{LIF}$  introduced in the beginning is generalized as follows:

**Definition 1.** If  $\mathcal{M}$  is a  $\mathcal{LIF}$ , then the *norm order* of  $\mathcal{M}$  is the quantity

$$\|\text{ord}\|_{\mathcal{M}} = \sup \left\{ \frac{1}{2} \|D^2f(0)\| : f \in \mathcal{M} \right\} = \alpha.$$

In [17, Theorem 3.1], it has been shown that  $\alpha \geq 1$ . As in the planar case, the universal linearly-invariant family  $\mathcal{M}_\alpha$  of order  $\alpha$  is defined as the union of all linearly invariant families of order less than or equal to  $\alpha$  (cf. [19]).

Our main aim of this paper is to extend the corresponding results of [23] and [24] to higher dimensional case.

## 2. MAIN RESULTS

Let  $\mathcal{PH}(\alpha, k)$  denote the set of all sense-preserving mappings  $f = h + \bar{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$  with the normalization  $h(0) = g(0) = 0$ ,  $\|Dh(0) + \overline{Dg(0)}\| = 1$ ,  $[Dh(0)]^{-1}h(z) \in \mathcal{M}_\alpha$ , and such that for  $k \in [0, 1)$ ,

$$\|Dg(z)[Dh(z)]^{-1}\| \leq k,$$

where  $h$  is locally biholomorphic and  $g$  is holomorphic in  $\mathbb{B}^n$ .

Obviously, if  $n = 1$ , then  $\mathcal{PH}(\alpha, k)$  coincides with the set  $H(\alpha, K)$  of [23] and [24]. As a generalization of [23, Theorem 1], we have.

**Theorem 1.** *For  $\alpha < \infty$ , the classes  $\mathcal{PH}(\alpha, k)$  are compact with respect to the topology of almost uniform convergence in  $\mathbb{B}^n$ .*

The derivative of  $f = h + \bar{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$  in the direction of vector  $\theta \in \partial\mathbb{B}^n$  at the point  $z$  will be denoted by

$$\partial_\theta f(z) = \lim_{\rho \rightarrow 0^+} \frac{f(z + \rho\theta) - f(z)}{\rho} = Dh(z)\theta + \overline{Dg(z)\theta},$$

where  $h$  and  $g$  are holomorphic in  $\mathbb{B}^n$ . We use the standard notations:

$$\Lambda_f = \max_{\theta \in \partial \mathbb{B}^n} \|\partial_\theta f\| \quad \text{and} \quad \lambda_f = \min_{\theta \in \partial \mathbb{B}^n} \|\partial_\theta f\|.$$

With this setting, we now present a generalization of [23, Theorem 2].

**Theorem 2.** *For  $\alpha < \infty$ , let  $f = h + \bar{g} \in \mathcal{PH}(\alpha, k)$ . Then*

$$(2.1) \quad \frac{1-k}{\|[Dh(0)]^{-1}\|} \frac{(1-\|z\|)^{\alpha-1}}{(1+\|z\|)^{\alpha+1}} \leq \Lambda_f(z) \leq \left( \frac{1+k}{1-k} \right) \frac{(1+\|z\|)^{\alpha-1}}{(1-\|z\|)^{\alpha+1}}$$

and

$$(2.2) \quad \|f(z)\| \leq \frac{1+k}{2\alpha(1-k)} \left\{ \frac{(1+\|z\|)^\alpha}{(1-\|z\|)^\alpha} - 1 \right\}.$$

*In particular, if  $n = 1$ , then the estimate of (2.1) is sharp for  $\theta = \pm \frac{\pi}{2}$ . Moreover, if  $z = re^{it}$ , then the equality on the right of (2.1) is obtained for  $f(z) = h(z) - k\bar{h}(\bar{z})$ ,*

$$h(z) = \frac{e^{it}}{2\alpha(1-k)} \left[ \left( \frac{1+ze^{-it}}{1-ze^{-it}} \right)^\alpha - 1 \right]$$

*and the equality on the left of (2.1) is obtained for  $f(z) = h^*(z) + k\overline{h^*(z)}$ ,*

$$h^*(z) = \frac{e^{it}}{2\alpha(1+k)} \left[ \left( \frac{1-ze^{-it}}{1+ze^{-it}} \right)^\alpha - 1 \right].$$

The following result is a covering theorem of  $\mathcal{PH}(\alpha, k)$ .

**Theorem 3.** *For  $r \in (0, 1]$  and  $\alpha < \infty$ , if  $f = h + \bar{g} \in \mathcal{PH}(\alpha, k)$ , then  $f(\mathbb{B}^n(r))$  contains a univalent ball  $\mathbb{B}^n(R)$  with*

$$R \geq \frac{(1-k)|\det Dh(0)|}{\|Dh(0)\|^{n-1}} \int_0^r \frac{(1-x)^{(2n-1)\alpha+(n-3)/2}}{(1+x)^{(2n-1)\alpha-(n-3)/2}} dx.$$

*In particular, if  $n = 1$ , then  $R = (1-k) \left[ 1 - \left( \frac{1-r}{1+r} \right)^\alpha \right] / [2\alpha(1+k)]$ , and the extreme function  $f = h + k\bar{h}$  shows that this estimate is sharp, where*

$$h(z) = \frac{\pm i}{2\alpha(1+k)} \left[ \left( \frac{1 \pm iz}{1 \mp iz} \right) - 1 \right].$$

We remark that Theorem 3 is a generalization of [23, Theorem 3].

**Theorem 4.** *For  $\alpha < \infty$ , if  $f = h + \bar{g} \in \mathcal{PH}(\alpha, k)$ , then*

$$|\det J_f(z)| \geq \frac{(1-k^2)^n}{(\det[Dh(0)]^{-1})^2} \frac{(1-\|z\|)^{2n\alpha-n-1}}{(1+\|z\|)^{2n\alpha+n+1}}.$$

For  $r \in (0, 1)$ , a univalent mapping  $f = h + \bar{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$  with  $h(0) = g(0) = 0$ ,  $Dg(0) = 0$  and

$$\|Dg[Dh]^{-1}\| < 1$$

is called *fully starlike* if it maps every ball  $\overline{\mathbb{B}^n(r)}$  onto a starlike domain with respect to the origin, where  $h$  is locally biholomorphic and  $g$  is holomorphic in  $\mathbb{B}^n$  (cf. [8]). The following result is a generalization of [5, Theorem 1.3].

**Theorem 5.** *Let  $r \in (0, 1)$  and  $f = h + \bar{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$  be fully starlike, where  $h$  is locally biholomorphic and  $g$  is holomorphic in  $\mathbb{B}^n$ . Then for all  $z \in \overline{\mathbb{B}^n(r)}$ ,*

$$\|h(z)\| \leq \frac{1}{1-r} \|f(z)\|.$$

Furthermore, if  $h \in \mathcal{M}_\alpha$ , then

(a) for  $z \in \mathbb{B}^n(r_0)$ ,

$$\|f(z)\| \geq r_0^2(1-r_0) \frac{\|z\|}{(r_0 + \|z\|)^2},$$

where  $r_0 = 4\alpha/(1+4\alpha^2)$ ;

(b)  $f$  differs from zero in  $\mathbb{B}^n(r_0) \setminus \{0\}$ .

We remark that

$$\frac{4\alpha}{1+4\alpha^2} = \frac{1}{\alpha} - \frac{1}{\alpha(1+4\alpha^2)} \sim \frac{1}{\alpha}$$

as  $\alpha \rightarrow \infty$ . Hence Theorem 5(b) is a generalization of [24, Theorem 1].

A continuous mapping  $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$  is called  $K$ -quasiregular if  $f \in W_{n,\text{loc}}^1(\Omega)$  and

$$\|Df(x)\|^n \leq K \det J_f(x) \text{ for almost every } x \in \Omega,$$

where  $K (\geq 1)$  is a constant. Here  $f \in W_{n,\text{loc}}^1(\Omega)$  means that the distributional derivatives  $\partial f_j / \partial x_k$  of the coordinates  $f_j$  of  $f$  are locally in  $L^n(\Omega)$  and  $J_f(x)$  denotes the Jacobian of  $f$  (cf. [26]).

Let  $f = (f_1 \cdots f_n)^T \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$ . For  $j \in \{1, \dots, n\}$ , we let  $z = (z_1 \cdots z_n)^T$ ,  $z_j = x_j + iy_j$  and  $f_j(z) = u_j(z) + iv_j(z)$ , where  $u_j$  and  $v_j$  are real pluriharmonic functions from  $\mathbb{B}^n$  into  $\mathbb{R}$ . We denote the real Jacobian matrix of  $f$  by

$$J_f = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial y_2} & \cdots & \frac{\partial u_1}{\partial x_n} & \frac{\partial u_1}{\partial y_n} \\ \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial y_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial y_2} & \cdots & \frac{\partial v_1}{\partial x_n} & \frac{\partial v_1}{\partial y_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial y_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial y_2} & \cdots & \frac{\partial u_n}{\partial x_n} & \frac{\partial u_n}{\partial y_n} \\ \frac{\partial v_n}{\partial x_1} & \frac{\partial v_n}{\partial y_1} & \frac{\partial v_n}{\partial x_2} & \frac{\partial v_n}{\partial y_2} & \cdots & \frac{\partial v_n}{\partial x_n} & \frac{\partial v_n}{\partial y_n} \end{pmatrix}.$$

Let  $\mathbb{B}_{\mathbb{R}}^{2n}$  denote the unit ball of  $\mathbb{R}^{2n}$ . Then

$$\Lambda_f = \max_{\theta \in \partial \mathbb{B}_{\mathbb{R}}^{2n}} \|J_f \theta\| \text{ and } \lambda_f = \min_{\theta \in \partial \mathbb{B}_{\mathbb{R}}^{2n}} \|J_f \theta\|.$$

**Theorem 6.** *Let  $f = h + \bar{g} \in \mathcal{PH}(\mathbb{B}^n, \mathbb{C}^n)$  with  $\|Dg(z)[Dh(z)]^{-1}\| \leq c < 1$  for  $z \in \mathbb{B}^n$ , where  $c$  is a positive constant. Then*

(a)  $f$  is a quasiregular mapping if and only if  $h$  is a quasiregular mapping;

Value of $n$	Value of $k_n$
1	0.423166
2	0.230006
3	0.157659
4	0.119898
5	0.0967215

TABLE 1. Values of  $k_n$  of Equation (2.3) for  $n = 1, 2, 3, 4, 5$ 

(b)  $f(\mathbb{B}^n)$  contains a univalent ball with the radius

$$R \geq \frac{k_n \pi}{8m} \left( \frac{k_n \pi \sqrt{1-c}}{4K \sqrt{1+c} \log(1/(1-k_n))} \right)^{4n-1},$$

where  $m \approx 4.2$ ,  $\det J_f(0) = 1$ ,  $h$  is a  $K$ -quasiregular mapping with  $K \geq 1$  and  $0 < k_n < 1$  is a unique root such that

$$(2.3) \quad -4n \log(1 - k_n) = (4n - 1) \frac{k_n}{1 - k_n}.$$

The roots  $k_n$  in  $(0, 1)$  of the equation (2.3) for the values of  $n = 1, 2, 3, 4, 5$  are listed in Table 1 for a ready reference.

The proofs of Theorems 1–6 will be presented in Section 3.

### 3. PROOFS OF THE MAIN THEOREMS

**Proof of Theorem 1.** Consider a sequence  $f_m = h_m + \bar{g}_m \in \mathcal{PH}(\alpha, k)$ . By definition, we have the conditions  $\|Dh_m(0) + \overline{Dg_m(0)}\| = 1$  and  $\|Dg_m(z)[Dh_m(z)]^{-1}\| \leq k$ , we see that

$$\|Dh_m(0)\| \leq 1 + \|Dg_m(0)\|$$

whereas the second condition gives

$$\|Dg_m(0)\| = \|Dg_m(0)[Dh_m(0)]^{-1}[Dh_m(0)]\| \leq k\|Dh_m(0)\|.$$

Using the last two inequalities, we easily have

$$(3.1) \quad \|Dg_m(0)\| \leq \frac{k}{1-k} \quad \text{and} \quad \|Dh_m(0)\| \leq \frac{1}{1-k}.$$

By (3.1),  $[Dh_m(0)]^{-1}h_m(z) \in \mathcal{M}_\alpha$  and thus by [17, Theorem 4.1], we obtain that

$$(3.2) \quad \frac{(1 - \|z\|)^{\alpha-1}}{(1 + \|z\|)^{\alpha+1}} \leq \|[Dh_m(0)]^{-1}Dh_m(z)\| \leq \frac{(1 + \|z\|)^{\alpha-1}}{(1 - \|z\|)^{\alpha+1}},$$

which implies

$$\begin{aligned} \|[Dh_m(z)]\| &= \|Dh_m(0)[Dh_m(0)]^{-1}Dh_m(z)\| \\ &\leq \|[Dh_m(0)]^{-1}Dh_m(z)\| \|Dh_m(0)\| \\ &\leq \frac{1}{(1-k)} \frac{(1 + \|z\|)^{\alpha-1}}{(1 - \|z\|)^{\alpha+1}}. \end{aligned}$$

Moreover, by the definition of  $\mathcal{PH}(\alpha, k)$ , it follows that

$$\|Dg_m(z)\| \leq k\|Dh_m(z)\| \leq \frac{k}{(1-k)} \frac{(1+\|z\|)^{\alpha-1}}{(1-\|z\|)^{\alpha+1}}.$$

Hence  $Dh_m(z)$  and  $Dg_m(z)$  are uniformly bounded in compact subsets of  $\mathbb{B}^n$ , which implies  $\mathcal{PH}(\alpha, k)$  are compact.  $\square$

**Proof of Theorem 2.** Let  $f = h + \bar{g} \in \mathcal{PH}(\alpha, k)$  for some  $\alpha < \infty$ . By the definition of directional derivatives, we have

$$\begin{aligned} \|\partial_\theta f(z)\| &= \left\| Dh(z)\theta + \overline{Dg(z)[Dh(z)]^{-1}Dh(z)\theta} \right\| \\ &\geq \|Dh(z)\theta\| (1 - \|Dg(z)[Dh(z)]^{-1}\|) \\ &\geq (1-k)\|Dh(z)\theta\| \end{aligned}$$

and similarly,

$$\begin{aligned} \|\partial_\theta f(z)\| &\leq \|Dh(z)\theta\| (1 + \|Dg(z)[Dh(z)]^{-1}\|) \\ &\leq (1+k)\|Dh(z)\theta\|. \end{aligned}$$

It follows that

$$(3.3) \quad (1-k)\|Dh(z)\| \leq \Lambda_f(z) = \max_{\theta \in \partial \mathbb{B}^n} \|\partial_\theta f(z)\| \leq (1+k)\|Dh(z)\|.$$

Again, by elementary calculations, we have

$$\|Dh(z)\| = \|Dh(0)[Dh(0)]^{-1}Dh(z)\| \leq \|[Dh(0)]^{-1}Dh(z)\| \|Dh(0)\|,$$

which gives

$$(3.4) \quad \frac{\|Dh(z)\|}{\|Dh(0)\|} \leq \|[Dh(0)]^{-1}Dh(z)\| \leq \|Dh(z)\| \|[Dh(0)]^{-1}\|.$$

By  $[Dh(0)]^{-1}h(z) \in \mathcal{M}_\alpha$  and [17, Theorem 4.1], we deduce that

$$(3.5) \quad \frac{(1-\|z\|)^{\alpha-1}}{(1+\|z\|)^{\alpha+1}} \leq \|[Dh(0)]^{-1}Dh(z)\| \leq \frac{(1+\|z\|)^{\alpha-1}}{(1-\|z\|)^{\alpha+1}}.$$

By (3.4) and (3.5), we get

$$(3.6) \quad \frac{1}{\|[Dh(0)]^{-1}\|} \frac{(1-\|z\|)^{\alpha-1}}{(1+\|z\|)^{\alpha+1}} \leq \|Dh(z)\| \leq \frac{(1+\|z\|)^{\alpha-1}}{(1-\|z\|)^{\alpha+1}} \|Dh(0)\|,$$

which implies

$$(3.7) \quad \frac{1-k}{\|[Dh(0)]^{-1}\|} \frac{(1-\|z\|)^{\alpha-1}}{(1+\|z\|)^{\alpha+1}} \leq \Lambda_f(z) \leq \frac{(1+\|z\|)^{\alpha-1}}{(1-\|z\|)^{\alpha+1}} \|Dh(0)\| (1+k).$$

Applying (3.7) and the inequality,

$$(3.8) \quad \frac{1}{1+k} \leq \|Dh(0)\| \leq \frac{1}{1-k},$$

we conclude that

$$(3.9) \quad \frac{1-k}{\|[Dh(0)]^{-1}\|} \frac{(1-\|z\|)^{\alpha-1}}{(1+\|z\|)^{\alpha+1}} \leq \Lambda_f(z) \leq \frac{1+k}{(1-k)} \frac{(1+\|z\|)^{\alpha-1}}{(1-\|z\|)^{\alpha+1}}.$$

Now we prove (2.2). Let  $[0, z]$  be the segment from 0 to  $z \in \mathbb{B}^n$ . Then by using (3.9), we have

$$\begin{aligned}
\|f(z)\| &= \left\| \int_{[0,z]} df(\zeta) \right\| = \left\| \int_{[0,z]} Dh(\zeta) d\zeta + \overline{Dg(\zeta) d\zeta} \right\| \\
&\leq \int_{[0,z]} \Lambda_f(\zeta) \|d\zeta\| \\
&= \frac{1+k}{1-k} \int_0^1 \frac{(1+t\|z\|)^{\alpha-1}}{(1-t\|z\|)^{\alpha+1}} \|z\| dt \\
&= \frac{1+k}{2\alpha(1-k)} \left\{ \frac{(1+\|z\|)^\alpha}{(1-\|z\|)^\alpha} - 1 \right\}.
\end{aligned}$$

The proof of this theorem is complete.  $\square$

**Lemma A.** ([15, Lemma 4]) *Let  $A$  be an  $n \times n$  complex (real) matrix with  $\|A\| \neq 0$ . Then for all unit vector  $\theta \in \partial\mathbb{B}^n$ , the inequality*

$$\|A\theta\| \geq \frac{|\det A|}{\|A\|^{n-1}}$$

*holds.*

**Proof of Theorem 3.** Let  $\rho$  be the radius of the largest univalence ball of center 0 and contained in  $f(\mathbb{B}^n(r))$ . Then we have  $\|f(z_0)\| = \rho$  for some  $z_0$  with  $\|z_0\| = r$ . Let  $[0, f(z_0)]$  denote the segment from 0 to  $f(z_0)$  and  $\gamma$  be a curve joining 0 and  $z_0$  in  $\mathbb{B}^n(r)$ , which is the preimage of  $[0, f(z_0)]$  for the mapping  $f$ . We use  $\gamma(t)$  to denote a smooth parametrization of  $\gamma$  with  $\gamma(0) = 0$  and  $\gamma(1) = z_0$ , where  $t \in [0, 1]$ .

Applying [17, Theorem 4.1 (4.2)] and Lemma A, we get

$$\begin{aligned}
\|\partial_\theta f(z)\| &= \left\| Dh(z)\theta + \overline{Dg(z)[Dh(z)]^{-1}Dh(z)\theta} \right\| \\
&\geq \|Dh(z)\theta\| (1 - \|Dg(z)[Dh(z)]^{-1}\|) \\
&\geq (1-k)\|Dh(z)\theta\| \\
&= (1-k) \left\| Dh(0) \frac{[Dh(0)]^{-1}Dh(z)\theta}{\|[Dh(0)]^{-1}Dh(z)\theta\|} \right\| \|[Dh(0)]^{-1}Dh(z)\theta\| \\
&\geq (1-k) \frac{(1-\|z\|)^{(2n-1)\alpha+(n-3)/2}}{(1+\|z\|)^{(2n-1)\alpha-(n-3)/2}} \min_{\xi \in \mathbb{B}^n} \|Dh(0)\xi\|
\end{aligned}$$



which implies that

$$\begin{aligned}
\rho &= |f(z_0)| = \left\| \int_0^1 \frac{d}{dt} f(\gamma(t)) dt \right\| \\
&= \int_0^1 \left\| \frac{d}{dt} f(\gamma(t)) \right\| dt = \int_0^1 \|\partial_\theta f(\gamma(t))\| |\gamma'(t)| dt \\
&\geq (1-k) \min_{\theta \in \mathbb{B}^n} \|Dh(\gamma(0))\theta\| \int_0^1 \frac{(1 - \|\gamma(t)\|)^{(2n-1)\alpha + (n-3)/2}}{(1 + \|\gamma(t)\|)^{(2n-1)\alpha - (n-3)/2}} \|d\gamma(t)\| \\
&\geq (1-k) \min_{\theta \in \mathbb{B}^n} \|Dh(0)\theta\| \int_0^r \frac{(1 - \|z\|)^{(2n-1)\alpha + (n-3)/2}}{(1 + \|z\|)^{(2n-1)\alpha - (n-3)/2}} d\|z\| \\
&\geq \frac{(1-k)|\det Dh(0)|}{\|Dh(0)\|^{n-1}} \int_0^r \frac{(1 - \|z\|)^{(2n-1)\alpha + (n-3)/2}}{(1 + \|z\|)^{(2n-1)\alpha - (n-3)/2}} d\|z\|,
\end{aligned}$$

where  $\gamma'(t) = |\gamma'(t)|\theta$ .

In particular, if  $n = 1$ , then

$$\begin{aligned}
\rho &\geq (1-k) \min_{\xi \in \mathbb{B}^n} \|Dh(0)\xi\| \int_0^r \frac{(1 - \|z\|)^{(2n-1)\alpha + (n-3)/2}}{(1 + \|z\|)^{(2n-1)\alpha - (n-3)/2}} d\|z\| \\
&\geq \frac{1-k}{1+k} \int_0^r \frac{(1-x)^{\alpha-1}}{(1+x)^{\alpha+1}} dx \\
&= \frac{1-k}{2\alpha(1+k)} \left[ 1 - \left( \frac{1-r}{1+r} \right)^\alpha \right].
\end{aligned}$$

The proof of the theorem is complete.  $\square$

**Lemma 1.** Suppose that  $A = (a_{ij})$  is an  $n \times n$  matrix. Then

$$\left( \min_{\theta \in \partial \mathbb{B}^n} \|A\theta\| \right)^n \leq |\det A| \leq \|A\|^n.$$

*Proof.* If  $A^* = (\overline{a_{ji}})$ , then the product  $A^*A$  is a positive semi-definite matrix. Let  $\lambda_1, \dots, \lambda_n$  ( $0 \leq \lambda_1 \leq \dots \leq \lambda_n$ ) be the  $n$  eigenvalues of the matrix  $A^*A$ . Then

$$\sqrt{\lambda_n} = \max\{\|A\theta\| : \theta \in \partial \mathbb{B}^n\} \quad \text{and} \quad \sqrt{\lambda_1} = \min\{\|A\theta\| : \theta \in \partial \mathbb{B}^n\},$$

which implies that

$$\|A\|^n \geq |\det A| = \sqrt{\prod_{k=1}^n \lambda_k} \geq (\sqrt{\lambda_1})^n = \left( \min_{\theta \in \partial \mathbb{B}^n} \|A\theta\| \right)^n.$$

The proof of the lemma is complete.  $\square$

**Proof of Theorem 4.** In view of Lemma 1 and [16, Theorem 5.1],  $J_f$  given by (1.1) shows that

$$\begin{aligned}
|\det J_f(z)| &= |\det Dh(z)|^2 \det \left( I_n - Dg(z)[Dh(z)]^{-1} \overline{Dg(z)[Dh(z)]^{-1}} \right) \\
&\geq |\det Dh(z)|^2 \min_{\theta \in \partial \mathbb{B}^n} \left\| \left( I_n - Dg(z)[Dh(z)]^{-1} \overline{Dg(z)[Dh(z)]^{-1}} \right) \theta \right\|^n \\
&\geq |\det Dh(z)|^2 \left( 1 - \|Dg(z)[Dh(z)]^{-1}\|^2 \right)^n \\
&\geq |\det Dh(z)|^2 (1 - k^2)^n \\
&= \frac{|\det ([Dh(0)]^{-1} Dh(z))|^2 (1 - k^2)^n}{(\det [Dh(0)]^{-1})^2} \\
&\geq \frac{(1 - k^2)^n}{(\det [Dh(0)]^{-1})^2} \frac{(1 - \|z\|)^{2n\alpha - n - 1}}{(1 + \|z\|)^{2n\alpha + n + 1}}.
\end{aligned}$$

The proof of the theorem is complete.  $\square$

**Proof of Theorem 5.** By the inverse mapping theorem, we know that  $f^{-1}$  is differentiable. Let  $f^{-1} = (\sigma_1 \cdots \sigma_n)^T$ . Then for  $j, m \in \{1, \dots, n\}$ , we use  $Df^{-1}$  and  $\overline{D}f^{-1}$  to denote the two  $n \times n$  matrices  $(\partial \sigma_j / \partial z_m)_{n \times n}$  and  $(\partial \sigma_j / \partial \bar{z}_m)_{n \times n}$ , respectively.

Differentiation of the equation  $f^{-1}(f(z)) = z$  yields the following relations

$$\begin{cases} Df^{-1}Dh + \overline{D}f^{-1}Dg = I_n, \\ Df^{-1}\overline{D}g + \overline{D}f^{-1}\overline{D}h = 0, \end{cases}$$

which give

$$(3.10) \quad \begin{cases} DhDf^{-1} = (I_n - \overline{D}g[\overline{D}h]^{-1}Dg[Dh]^{-1})^{-1}, \\ Dh\overline{D}f^{-1} = -(I_n - \overline{D}g[\overline{D}h]^{-1}Dg[Dh]^{-1})^{-1}\overline{D}g[\overline{D}h]^{-1}. \end{cases}$$

By (3.10), we get

$$\begin{aligned}
\|DhDf^{-1}\| + \|Dh\overline{D}f^{-1}\| &= \left\| (I_n - \overline{D}g[\overline{D}h]^{-1}Dg[Dh]^{-1})^{-1} \right\| \\
&\quad + \left\| (I_n - \overline{D}g[\overline{D}h]^{-1}Dg[Dh]^{-1})^{-1} \overline{D}g[\overline{D}h]^{-1} \right\| \\
&\leq \left\| (I_n - \overline{D}g[\overline{D}h]^{-1}Dg[Dh]^{-1})^{-1} \right\| (1 + \|Dg[Dh]^{-1}\|) \\
&\leq \frac{1 + \|Dg[Dh]^{-1}\|}{1 - \|\overline{D}g[\overline{D}h]^{-1}Dg[Dh]^{-1}\|} \\
(3.11) \quad &\leq \frac{1 + \|Dg[Dh]^{-1}\|}{1 - \|Dg[Dh]^{-1}\|^2} = \frac{1}{1 - \|Dg[Dh]^{-1}\|}.
\end{aligned}$$

Since  $\Omega = f(\overline{\mathbb{B}^n(r)})$  is starlike, for each point  $z_0 \in \overline{\mathbb{B}^n(r)}$  and  $t \in [0, 1]$ , we have  $\varphi(t) = tf(z_0) \in \Omega$ , where  $f = (f_1 \cdots f_n)^T$ . Let  $\gamma = f^{-1} \circ \varphi$ . For any fixed  $\theta \in \partial \mathbb{B}^n$ , let  $A_\theta = Dg[Dh]^{-1}\theta$ . By Schwarz's lemma, for  $z \in \mathbb{B}^n(r)$ ,  $\|A_\theta(z)\| \leq \|z\|$  if  $r \in (0, 1)$ . The arbitrariness of  $\theta \in \partial \mathbb{B}^n$  gives

$$(3.12) \quad \|Dg(z)[Dh(z)]^{-1}\| \leq \|z\| \leq r$$

for  $z \in \mathbb{B}^n(r)$ . As before, by (3.11) and (3.12), we obtain that

$$\begin{aligned}
\|h(z_0)\| &= \left\| \int_0^1 Dh(\gamma(t)) \frac{d}{dt} \gamma(t) dt \right\| \\
&= \left\| \int_0^1 Dh(\gamma(t)) \left[ Df^{-1}(\varphi(t)) D\varphi(t) + \overline{Df^{-1}(\varphi(t))} \overline{D\varphi(t)} \right] dt \right\| \\
&\leq \int_0^1 (\|Dh(\gamma(t)) Df^{-1}(\varphi(t))\| + \|Dh(\gamma(t)) \overline{Df^{-1}(\varphi(t))}\|) \|D\varphi(t)\| dt \\
&\leq \|f(z_0)\| \int_0^1 (1 + \|Dg(\gamma(t)) [Dh(\gamma(t))]^{-1}\|) \\
&\quad \times \left\| I_n - \overline{Dg(\gamma(t))} [\overline{Dh(\gamma(t))}]^{-1} Dg(\gamma(t)) [Dh(\gamma(t))]^{-1} \right\| dt \\
&\leq \int_0^1 \frac{1 + \|Dg(\gamma(t)) [Dh(\gamma(t))]^{-1}\|}{1 - \left\| \overline{Dg(\gamma(t))} [\overline{Dh(\gamma(t))}]^{-1} Dg(\gamma(t)) [Dh(\gamma(t))]^{-1} \right\|} dt \\
&\quad \times \|f(z_0)\| \\
&\leq \|f(z_0)\| \int_0^1 \frac{1}{1 - \|Dg(\gamma(t)) [Dh(\gamma(t))]^{-1}\|} dt \\
&\leq \frac{1}{1-r} \|f(z_0)\|,
\end{aligned}$$

where

$$D\varphi(t) = \begin{pmatrix} f_1(z_0) & 0 & 0 & \cdots & 0 \\ 0 & f_2(z_0) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & f_{n-1}(z_0) & 0 \\ 0 & 0 & \cdots & 0 & f_n(z_0) \end{pmatrix}$$

is a diagonal matrix.

Now we prove the second part of Theorem 5(a) and (b). By [17, Theorem 5.7], we know that  $h(\mathbb{B}^n(r_0))$  is starlike. For  $\zeta \in \mathbb{B}^n$ , let  $H(\zeta) = h(r_0\zeta)/r_0$ . Applying [1, Theorem 2.1] to  $H$ , we know that for  $\zeta \in \mathbb{B}^n$ ,

$$\|H(\zeta)\| \geq \frac{\|\zeta\|}{(1 + \|\zeta\|)^2},$$

which implies for  $z \in \mathbb{B}^n(r_0)$ ,

$$(3.13) \quad \|h(z)\| \geq \frac{r_0^2 \|z\|}{(r_0 + \|z\|)^2}.$$

Then Theorem 5 (a) follows from (3.13), and Theorem 5 (b) easily follows from Theorem 5(a). The proof of the theorem is complete.  $\square$

**Proof of Theorem 6.** We first prove the sufficiency of part (a). Without loss of generality, we assume that

$$(3.14) \quad \|Dh(z)\| \leq K |\det Dh(z)|^{\frac{1}{n}} \quad \text{for } z \in \mathbb{B}^n,$$

where  $K \geq 1$  is a constant.

As in the proof of Theorem 4, (3.14) and Lemma 1, for  $z \in \mathbb{B}^n$ , we have

$$|\det J_f(z)| \geq |\det Dh(z)|^2 (1 - c^2)^n$$

so that

$$|\det Dh(z)|^{\frac{1}{n}} \leq \frac{|\det J_f(z)|^{\frac{1}{2n}}}{\sqrt{1 - c^2}}.$$

Moreover,

$$\Lambda_f(z) = \max_{\theta \in \partial \mathbb{B}_{\mathbb{R}}^{2n}} \|J_f(z)\theta\| \leq \|Dh(z)\| (1 + \|Dg(z)[Dh(z)]^{-1}\|) \leq \|Dh(z)\|(1 + c),$$

which by the last inequality gives that

$$(3.15) \quad \Lambda_f(z) \leq K \sqrt{\frac{1+c}{1-c}} |\det J_f(z)|^{\frac{1}{2n}}$$

and hence,  $f$  is a quasiregular mapping.

Next we prove the necessity of part (a). We assume that for  $z \in \mathbb{B}^n$ ,

$$(3.16) \quad \Lambda_f(z) \leq K_1 |\det J_f(z)|^{\frac{1}{n}},$$

where  $K_1 \geq 1$  is a constant.

As in the proof of Theorem 4, for  $z \in \mathbb{B}^n$ , by calculations and Lemma 1, we get

$$\begin{aligned} |\det J_f(z)| &= |\det Dh(z)|^2 \left| \det \left( I_n - Dg(z)[Dh(z)]^{-1} \overline{Dg(z)[Dh(z)]^{-1}} \right) \right| \\ &\leq |\det Dh(z)|^2 \left\| I_n - Dg(z)[Dh(z)]^{-1} \overline{Dg(z)[Dh(z)]^{-1}} \right\|^n \\ &\leq |\det Dh(z)|^2 (1 + c^2)^n \end{aligned}$$

so that

$$|\det Dh(z)|^{\frac{1}{n}} \geq \frac{|\det J_f(z)|^{\frac{1}{2n}}}{\sqrt{1 + c^2}}.$$

Furthermore,

$$\Lambda_f(z) = \max_{\theta \in \partial \mathbb{B}_{\mathbb{R}}^{2n}} \|J_f(z)\theta\| \geq \|Dh(z)\| (1 - \|Dg(z)[Dh(z)]^{-1}\|) \geq \|Dh(z)\|(1 - c),$$

which, by (3.16), implies that

$$\|Dh(z)\|(1 - c) \leq \Lambda_f(z) \leq K_1 |\det J_f(z)|^{\frac{1}{n}} \leq K_1 \sqrt{1 + c^2} |\det Dh(z)|^{\frac{1}{n}}.$$

Hence

$$\|Dh(z)\| \leq \frac{K_1 \sqrt{1 + c^2}}{1 - c} |\det Dh(z)|^{\frac{1}{n}},$$

which shows that  $h$  is a quasiregular mapping.

Now we prove part (b). By (3.15), we know that  $f$  is a pluriharmonic  $K_2$ -quasiregular mapping, where  $K_2 = K\sqrt{\frac{1+c}{1-c}}$ . Applying [3, Theorem 6], we know that  $f(\mathbb{B}^n)$  contains a univalent ball with the radius  $R$  with

$$R \geq \frac{k_n \pi}{8m} \left( \frac{k_n \pi}{4K_2 \log(1/(1 - k_n))} \right)^{4n-1},$$

where  $m \approx 4.2$  and  $0 < k_n < 1$  is a unique root such that

$$4n \log \frac{1}{1 - k_n} = (4n - 1) \frac{k_n}{1 - k_n}.$$

The proof of the theorem is complete.  $\square$

**Acknowledgements.** The research of the second author was supported by the project RUS/RFBR/P-163 under Department of Science & Technology and Russian Foundation for Basic Research and this author is currently on leave from the Department of Mathematics, Indian Institute of Technology Madras, Chennai-600 036, India.

## REFERENCES

1. R. W. BARNARD, C. H. FITZGERALD AND S. GONG, The growth and  $1/4$ -theorems for starlike mappings in  $\mathbb{C}^n$ , *Pacific J. Math.*, **150**(1991), 13–22.
2. D. M. CAMPBELL, Locally univalent function with locally univalent derivatives, *Trans. Amer. Math. Soc.* **162**(1971), 395–409.
3. H. CHEN AND P. M. GAUTHIER, The Landau theorem and Bloch theorem for planar harmonic and pluriharmonic mappings, *Proc. Amer. Math. Soc.*, **139**(2011), 583–595.
4. SH. CHEN, S. PONNUSAMY AND X. WANG, Equivalent moduli of continuity, Bloch’s theorem for pluriharmonic mappings in  $\mathbb{B}^n$ , *Proc. Indian Acad. Sci. (Math. Sci.)*, **122**(2012), 583–595.
5. SH. CHEN, S. PONNUSAMY AND X. WANG, Covering and distortion theorems for planar harmonic univalent mappings, *Arch. Math. (Basel)*, **101**(2013), 285–291.
6. SH. CHEN, S. PONNUSAMY AND X. WANG, The isoperimetric type and Fejer-Riesz type inequalities for pluriharmonic mappings, *Sci. Sin. Math.* (in Chinese), **44**(2014), 127–138.
7. SH. CHEN, S. PONNUSAMY AND X. WANG, Univalence criteria and Lipschitz-type spaces on pluriharmonic mappings, *Math. Scand.*, to appear.
8. M. CHUAQUI, P. DUREN AND B. OSGOOD, Curvature properties of planar harmonic mappings, *Comput. Methods Funct. Theory*, **4**(2004), 127–142.
9. J. G. CLUNIE AND T. SHEIL-SMALL, Harmonic univalent functions, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **9**(1984), 3–25.
10. P. DUREN, *Harmonic Mappings in the Plane*, Cambridge Univ. Press, 2004.
11. P. DUREN, H. HAMADA AND G. KOHR, Two-point distortion theorems for harmonic and pluriharmonic mappings, *Trans. Amer. Math. Soc.*, **363**(2011), 6197–6218.
12. A. J. IZZO, Uniform algebras generated by holomorphic and pluriharmonic functions, *Trans. Amer. Math. Soc.*, **339**(1993), 835–847.
13. W. KOEPF, close-to-convex functions and linear-invariant families, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **8**(1983), 349–355.
14. H. LEWY, On the non-vanishing of the Jacobian in certain one-to-one mappings, *Bull. Amer. Math. Soc.*, **42**(1936), 689–692.
15. X. Y. LIU, Bloch functions of several complex variables, *Pacific J. Math.*, **152**(1992), 347–363.
16. J. A. PFALTZGRAFF, Distortion of locally biholomorphic maps of the  $n$ -ball, *Complex Variables* **33**(1997), 239–253.

17. J. A. PFALTZGRAFF AND T. J. SUFFRIDGE, Norm order and geometric properties of holomorphic mappings in  $\mathbb{C}^n$ , *J. Anal. Math.*, **82**(2000), 285–313.
18. J. A. PFALTZGRAFF AND T. J. SUFFRIDGE, Linear invariance, order and convex maps in  $C^n$ , *Complex Variables* **40**(1999), 35–50.
19. CH. POMMERENKE, Linear-invariante familien analytischer funktionen. I, *Math. Ann.*, **155**(1964), 108–154.
20. W. RUDIN, *Function theory in the unit ball of  $\mathbb{C}^n$* , Springer-Verlag, New York, Heidelberg, Berlin, 1980.
21. V. V. STARKOV, A theorem of regularity in universal linearly invariant families of functions, *Proceedings of the International Conference of Constructed Theory of Functions*, Varna 1984 (Sofia, 1984), 76–79.
22. V. V. STARKOV, Regularity theorems for universal linearly invariant families of functions, *Serdika* **11**(1985), 299–318.
23. V. V. STARKOV, Harmonic locally quasiconformal mappings, *Annales Universitatis Mariae Curie-Sklodowska. Sectio A. Mathematica*, **14**(1995), 183–197.
24. V. V. STARKOV, Univalence disks of harmonic locally quasiconformal mappings and harmonic Bloch functions, *Siberian Math. J.* **38**(1997), 791–800.
25. V. S. VLADIMIROV, *Methods of the Theory of Functions of Several Complex Variables*, (in Russian), M. I. T. Press, Cambridge, Mass., 1966.
26. M. VUORINEN, Conformal geometry and quasiregular mappings, *Lecture Notes in Math.* Vol. 1319, Springer-Verlag, 1988.

SH. CHEN, DEPARTMENT OF MATHEMATICS AND COMPUTATIONAL SCIENCE, HENGYANG NORMAL UNIVERSITY, HENGYANG, HUNAN 421008, PEOPLE'S REPUBLIC OF CHINA.

*E-mail address:* mathechen@126.com

S. PONNUSAMY, INDIAN STATISTICAL INSTITUTE (ISI), CHENNAI CENTRE, SETS (SOCIETY FOR ELECTRONIC TRANSACTIONS AND SECURITY), MGR KNOWLEDGE CITY, CIT CAMPUS, TARAMANI, CHENNAI 600 113, INDIA.

*E-mail address:* samy@isichennai.res.in, samy@iitm.ac.in